A linear dimensionless bound for the weighted Riesz vector

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Abstract

We show that the norm of the vector of Riesz transforms as operator in the weighted Lebesgue space L^2_ω is bounded by a constant multiple of the first power of the Poisson-A_2 characteristic of ω. The bound is free of dimension. We also show that for $n > 1$, the Poisson-A_2 class is properly included in the classical A_2 class.

Key words: Bellman function, Riesz transforms, weighted estimates

1 Introduction

A weight is a positive L^1_{loc} function. Muckenhoupt proved in [11] that for $1 < p < \infty$ the maximal function is bounded on $L^p(\omega)$ iff the weight ω belongs to the class A_p, where

$$\omega \in A_p \text{ iff } Q_p(\omega) := \sup_B \langle \omega \rangle_B \langle \omega^{-1/(p-1)} \rangle_B^{p-1} < \infty.$$

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Here the notation $\langle \cdot \rangle_B$ denotes the average over the ball B and the supremum runs over all balls B. Hunt, Muckenhoupt, Wheeden proved in [10] that the A_p condition also characterizes the boundedness of the Hilbert transform

$$Hf(x) = \frac{1}{\pi} \int \frac{f(y)}{x-y} dy$$

in $L^p(\omega)$. Coifman and Fefferman in [1] extended the theory to general Calderón-Zygmund operators.

The focus in this note is on the Riesz transforms in weighted spaces. The linear and optimal bound in terms of the classical A_2 characteristic has been established by one of the authors in [13]. The same estimate holds for all Calderón-Zygmund operators by Hytonen's theorem [5]. This remarkable result has been reproven by Lerner [9]. The bound depends upon the dimension in all these proofs, as one should expect.

We are interested in a version of A_2 which is particularly well-suited for working with the Riesz transforms in $\mathbb{R}^n$, where we exploit the intimate connection of Riesz transforms and harmonic functions. We use the Poisson-A_2 class, which considers Poisson averages instead of box averages in the definition of A_2 and obtain a bound free of dimension for the Riesz vector. This way of measuring the characteristic of the weight arises naturally when working with Bellman functions, when convexity is replaced by harmonicity. This was also the approach in [12] as well as [15] for the weighted Hilbert transform and [3] for unweighted Riesz transforms. From a probabilistic angle, the Poisson-A_2 characteristic also arises naturally as a consequence of the use of martingales driven by space-time Brownian motion as in Gundy-Varopoulos [4].

An altered version of one-dimensional Poisson extensions of weights made a reappearance in the works concerned with the famous two-weight problem for the Hilbert transform, see [6] and [7]. It enjoys its interpretation as a 'tamed' Hilbert transform, a feature that appears to be lost in higher dimensions. In the one-dimensional case, we see a quadratic relation between the Poisson characteristic and the classical characteristic, but the classes themselves are the same. Interestingly, these different A_2 classes are not identical when the dimension is larger. We will show examples of A_2 weights whose Poisson integral diverges when the dimension is at least two. Such weights belong to A_2 but not to Poisson-A_2. This shows that the Poisson characteristic used on a pair of weights such as for the two-weight problems, is not necessary in higher dimension. This is one of several obstacles when considering the two-weight question for the Riesz transforms, that is currently under investigation. We mention the recent advance [8] where the Poisson characteristic is modified.

2 Notation

The Riesz transforms R_k in $\mathbb{R}^n$ are the component operators of the Riesz vector $\vec{R}$, defined on the Schwartz class by

$$(\hat{R_k f})(\xi) = i\frac{\xi_k}{\|\xi\|}\hat{f}(\xi).$$

We consider the space L^2_ω, where ω is a positive scalar valued L^1_{loc} function, called a weight. More specifically, the space $L^2_\omega(\mathbb{R}^n;\mathbb{C})$ consists of all measurable functions $f : \mathbb{R}^n \to \mathbb{C}$ so that the quantity

$$\|f\|_\omega := \left(\int_{\mathbb{R}^n} |f(x)|^2\omega(x)dx\right)^{1/2}$$

is finite, where dx denotes the Lebesgue measure on $\mathbb{R}^n$. For the space of vector valued functions $L^2_\omega(\mathbb{R}^n;\mathbb{C}^n)$, we replace $|\cdot|$ by the ℓ^2 norm $\|\cdot\|$.

We are concerned with a special class of weights, called Poisson-A_2. We say $\omega \in \tilde{A}_2$ if

$$\tilde{Q}_2(\omega) := \sup_{(x,t)\in R^n\times R_+} P_t(\omega)(x)P_t(\omega^{-1})(x) < \infty \tag{1}$$

where P_t denotes the Poisson extension operator into the upper half space defined by

$$P_t = e^{-tA}$$

where we define $A := \sqrt{-\Delta}$ and Δ be the Laplacian in $\mathbb{R}^n$. The scalar Riesz transforms can be written as

$$R_k = \partial_k \circ A^{-1}.$$

The Poisson kernel has the form

$$P_t(y) = c_n\frac{t}{(t^2+|y|^2)^{\frac{n+1}{2}}}$$

where c_n is its normalizing factor. The extension operator becomes

$$P_t f(x) = c_n\int_{\mathbb{R}^n} f(y)P_t(x-y)dy.$$

3 Main results

The main purpose of this text is to provide the dimensionless estimate:

Theorem 3.1 *There exists a constant c that does not depend on the dimension n or the weight ω so that for all weights $\omega \in \tilde{A}_2$ the Riesz vector as an operator in weighted space $L^2_\omega \to L^2_\omega$ has operator norm $\|\vec{R}\|_{L^2_\omega \to L^2_\omega} \le c\tilde{Q}_2(\omega)$.*

We also investigate the relationship between different Muckenhoupt classes. Notably, their relation changes with dimension:

Theorem 3.2 *Poisson-A_2 and classical A_2 only define the same classes of weights when the dimension is one: $\tilde{A}_2 = A_2$ if and only if $n = 1$.*

4 The dimension-free estimate

Since

$$\|\vec{R}\|_{L^2_\omega(\mathbb{R}^n;\mathbb{C})\to L^2_\omega(\mathbb{R}^n;\mathbb{C}^n)} = \|\omega^{1/2}\vec{R}\omega^{-1/2}\|_{L^2(\mathbb{R}^n;\mathbb{C})\to L^2(\mathbb{R}^n;\mathbb{C}^n)}$$

where the outer multiplication by $\omega^{1/2}$ is a scalar multiplication. We can estimate $\|\vec{R}\|_{L^2_\omega\to L^2_\omega}$ via L^2 duality. It is sufficient to estimate

$$|(\vec{g}, \omega^{1/2}\vec{R}\omega^{-1/2}f)| \le c\tilde{Q}_2(\omega)\|f\|\|\vec{g}\|$$

for test functions (smooth and compactly supported) $f, \vec{g}$, where f is scalar valued and $\vec{g}$ vector valued. Or (considering $\omega^{-1/2}f$ instead of f and $\omega^{1/2}\vec{g}$ instead of $\vec{g}$):

$$|(\vec{g}, \vec{R}f)| \le c\tilde{Q}_2(\omega)\|\vec{g}\|_{\omega^{-1}}\|f\|_\omega.$$

To prove this estimate, we prove the following theorem:

Theorem 4.1 *For test functions $f, \vec{g}$ and $\omega \in \tilde{A}_2$ we have the following estimate:*

$$|(\vec{g}, \vec{R}f)| \le c\tilde{Q}_2(\omega)(\|\vec{g}\|^2_{\omega^{-1}} + \|f\|^2_\omega); \tag{2}$$

here c does not depend on $f, \vec{g}, n, k$ or ω.

Considering λf and $\lambda^{-1}\vec{g}$ for appropriate λ, with the considerations above yields Theorem 3.1.

Before we turn to the proof of Theorem 4.1, let us formulate several useful lemmata.

4.1 Three useful Lemmata

The following is a well known fact. It is, for example, stated in [4].

Lemma 4.2

$$(g, R_k f) = 4\int_0^\infty \left(\frac{d}{dt}P_t g, \partial_k P_t f\right) tdt.$$

The proof using semigroups is very simple and concise, so we include it for the convenience of the reader. Instead of using semigroups, the same result can be obtained by the use of the Fourier transform.

Proof. Observe that $F(0) = \int_0^\infty F''(t)tdt$ for sufficiently fast decaying F. So

$$(g, R_k f) = (P_0 g, P_0 R_k f) = \int_0^\infty \frac{d^2}{dt^2}(P_t g, P_t R_k f) \ \ tdt.$$

The right hand side becomes

$$\int_0^\infty \left((\frac{d^2}{dt^2}P_t g, P_t R_k f) + 2(\frac{d}{dt}P_t g, \frac{d}{dt}P_t R_k f) + (P_t g, \frac{d^2}{dt^2}P_t R_k f) \right) tdt.$$

Now we use the fact that $\frac{d}{dt}P_t = -AP_t$ and $\frac{d^2}{dt^2}P_t = A^2 P_t$ and symmetry of A to see that the above equals

$$4\int_0^\infty (AP_t g, AP_t R_k f)tdt.$$

Using the fact that $P_t g$ is harmonic we can replace A by $-\partial_t$ in this expression. Furthermore, observe that A commutes with P_t and ∂_k and that $R_k = \partial_k \circ A^{-1}$. We obtain

$$(g, R_k f) = 4\int_0^\infty (\frac{d}{dt}P_t g, \partial_k P_t f)tdt.$$

For function f and vector function $\vec{g}$ this becomes

$$(\vec{g}, \vec{R}f) = 4\int_0^\infty (\frac{d}{dt}P_t \vec{g}, \nabla P_t f)tdt.$$

QED

We will need the following form of a Lemma that has been proven in [3] and generalised in [2], the so-called 'ellipse lemma'.

Lemma 4.3 *Let $m, l, k \in \mathbb{N}$. Denote $d = m + l + k$. For arbitrary $u \in \mathbb{R}^{m+l+k}$ write $u = u_m \oplus u_l \oplus u_k$, where $u_i \in \mathbb{R}^i$ for $i = m, l, k$. Let $r_m = \|u_m\|, r_l = \|u_l\|$. Suppose the matrix $A \in \mathbb{R}^{d\times d}$ is such that*

$$(Au, u) \geq 2r_m r_l$$

for all $u \in \mathbb{R}^d$. Then there exists $\tau > 0$ so that

$$(Au, u) \geq \tau r_m^2 + \tau^{-1} r_l^2$$

for all $u \in \mathbb{R}^d$.

We will be using this lemma for $m = 2$, $l = 2n$ and $k = 4$.

Finally we crucially need the Lemma below, that has been proven in [14]. It was deduced from the sharp weighted inequality for martingales from [16].

Lemma 4.4 *For any* $Q > 1$ *let* $\mathcal{D}$ *be a subset of* $\mathbb{R} \times \mathbb{R} \times \mathbb{C} \times \mathbb{C}^n \times \mathbb{R} \times \mathbb{R}$

$$\mathcal{D}_Q = \{(\mathbf{X}, \mathbf{Y}, \mathbf{x}, \mathbf{y}, \mathbf{r}, \mathbf{s}) : |\mathbf{x}|^2 < \mathbf{X}\mathbf{s},\ \|\mathbf{y}\|^2 < \mathbf{Y}\mathbf{r},\ 1 < \mathbf{r}\mathbf{s} < Q\}.$$

For any compact $K \subset \mathcal{D}_Q$ *there exists an infinitely differentiable function* $B_{K,Q}$ *defined in a small neighborhood of* K *that still lies inside* $\mathcal{D}_Q$ *so that the following estimates hold in* K.

$$0 \leq B_{K,Q} \leq 5Q(\mathbf{X} + \mathbf{Y}), \tag{3}$$

$$-d^2 B_{K,Q} \geq 2|d\mathbf{x}|\|d\mathbf{y}\|. \tag{4}$$

The last inequality describes an operator inequality where the left hand side is the negative Hessian of B. This function is a mollified version of the Bellman function one gets by knowing that the equivalent weighted estimate holds for a certain dyadic model operator.

4.2 Proof of the main theorem

Recall the inequality of theorem (4.1). We want to show for test functions f and g and $\omega \in \tilde{A}_2$:

$$|(\vec{g}, \vec{R}f)| \leq c\tilde{Q}_2(\omega)(\|\vec{g}\|^2_{\omega^{-1}} + \|f\|^2_\omega).$$

For a fixed weight ω we let $Q = \tilde{Q}_2(\omega)$. This gives rise to the set $\mathcal{D}_Q$. We define

$$b_{K,Q}(x, t) = B_{K,Q}(v(x, t))$$

where

$$v(x, t) = \Big(P_t(|f|^2\omega), P_t(\|\vec{g}\|^2\omega^{-1}), P_t(f), P_t(\vec{g}), P_t(\omega), P_t(\omega^{-1})\Big)(x)$$

Here K is a compact subset of $\mathcal{D}_Q$ to be chosen later.

Note that the vector $v \in \mathcal{D}_Q$ for any choice of (x, t). This is ensured by $Q = \tilde{Q}_2(\omega)$ and several applications of Jensen's inequality. Notice also that the vector v takes compacts inside the interior of $\mathbb{R}^{n+1}_+$ to compacts K inside $\mathcal{D}_Q$

for fixed $f, \vec{g}, \omega$. By elementary application of the chain rule (using harmonicity of the components of v) one shows that

$$\Delta_{x,t} b(x,t) = \sum_{i=1}^{n} \left(d^2 B(v) \frac{\partial}{\partial x_i} v, \frac{\partial}{\partial x_i} v \right) + \left(d^2 B(v) \frac{\partial}{\partial t} v, \frac{\partial}{\partial t} v \right).$$

Here $\Delta_{x,t}$ is the full Laplacian in the upper half space

$$\Delta_{x,t} = \sum_{i=1}^{n} \partial_{x_i}^2 + \partial_t^2.$$

Notice that condition 4 means that at any $v = (\mathbf{X}, \mathbf{Y}, \mathbf{x}, \mathbf{y}, \mathbf{r}, \mathbf{s}) \in K \subset \mathcal{D}_Q$ we have the operator inequality for any $u \in \mathbb{R} \times \mathbb{R} \times \mathbb{C} \times \mathbb{C}^n \times \mathbb{R} \times \mathbb{R}$

$$\left(-d^2 B_{K,Q}(v) u, u \right) \geq 2(|d\mathbf{x}| \|d\mathbf{y}\| u, u) = 2|u_3| \|u_4\|.$$

In our situation, $f, \vec{g}, \omega$ and Q are fixed, but we have varying K, x, t. So Lemma 4.3 guarantees us existence of $\tau_{x,t,K}$ so that

$$\left(-d^2 B(v) \frac{\partial}{\partial x_i} v, \frac{\partial}{\partial x_i} v \right) \geq \tau_{x,t,K} |\frac{\partial}{\partial x_i} P_t f|^2 + \tau_{x,t,K}^{-1} \|\frac{\partial}{\partial x_i} P_t \vec{g}\|^2$$

for all i and

$$\left(-d^2 B(v) \frac{\partial}{\partial t} v, \frac{\partial}{\partial t} v \right) \geq \tau_{x,t,K} |\frac{\partial}{\partial t} P_t f|^2 + \tau_{x,t,K}^{-1} \|\frac{\partial}{\partial t} P_t \vec{g}\|^2.$$

So

$$\begin{aligned}
&-\Delta_{x,t} b_{K,Q}(x,t) \\
&\geq \tau_{x,t,K} \left(\sum_{i=1}^{n} |\frac{\partial}{\partial x_i} P_t f|^2 + |\frac{\partial}{\partial t} P_t f|^2 \right) \\
&\quad + \tau_{x,t,K}^{-1} \left(\sum_{i=1}^{n} \|\frac{\partial}{\partial x_i} P_t \vec{g}\|^2 + \|\frac{\partial}{\partial t} P_t \vec{g}\|^2 \right) \\
&\geq 2 \left(\sum_{i=1}^{n} |\frac{\partial}{\partial x_i} P_t f|^2 + |\frac{\partial}{\partial t} P_t f|^2 \right)^{1/2} \left(\sum_{i=1}^{n} \|\frac{\partial}{\partial x_i} P_t \vec{g}\|^2 + \|\frac{\partial}{\partial t} P_t \vec{g}\|^2 \right)^{1/2} \\
&\geq 2 \left(\sum_{i=1}^{n} |\frac{\partial}{\partial x_i} P_t f|^2 \right)^{1/2} \|\frac{\partial}{\partial t} P_t \vec{g}\| \\
&= 2 \|\nabla P_t f\| \|\frac{\partial}{\partial t} P_t \vec{g}\|
\end{aligned}$$

In what follows we first use formula 4.2, then the estimate for the Laplacian we just proved:

$$
\begin{aligned}
&|(\vec{g}, \vec{R}f)| \\
&\leq 4\int_0^\infty |\left(\frac{\partial}{\partial t}P_t\vec{g}, \nabla P_t f\right)|tdt \\
&\leq 4\int_0^\infty \int_{\mathbb{R}^n} \|\frac{\partial}{\partial t}P_t\vec{g}\|\|\nabla P_t f\|dxtdt \\
&\leq 2\int_0^\infty \int_{\mathbb{R}^n} -\Delta_{x,t}b_{K,Q}(x,t)dxtdt.
\end{aligned}
$$

It remains to see that

$$
-\int_0^\infty \int_{\mathbb{R}^n} \Delta_{x,t}b_{K,Q}(x,t)dxtdt \leq C\tilde{Q}_2(\|f\|^2_{\omega^{-1}} + \|\vec{g}\|^2_\omega) \tag{5}
$$

with C independent of n. In order to obtain this last estimate, we will apply Green's formula as well as some properties of our Bellman function. We are going to pass through values of the function b.

Recall the statement of Green's formula:

Theorem 4.5

$$
\int_\Omega f(x)\Delta g(x) - g(x)\Delta f(x)dA(x) = \int_{\partial\Omega}\left(f(t)\frac{\partial g}{\partial n}(t) - g(t)\frac{\partial f}{\partial n}(t)\right)dS(t)
$$

where n is the outward normal and dS surface measure on $\partial\Omega$.

In order to be accurate, we are obliged to take care of a few technicalities first.

Let T_R be a cylinder with square base in upper half space $[-R,R]^n \times [0,2R]$. For R not too small, the point $(0,1)$ lies inside T_R. Let $T_{R,\epsilon} = T_R + (0,\epsilon)$. For any interior point (ξ,τ), let $G^{R,\epsilon}[(x,t),(\xi,\tau)]$ be its Green function, meaning that

$$
\Delta_{x,t}G^{R,\epsilon}[(x,t),(\xi,\tau)] = -\delta_{(\xi,\tau)} \quad \text{and} \quad G^{R,\epsilon} = 0 \quad \text{on} \quad \partial T_{R,\epsilon}.
$$

Notice that $RT_{1,0} = T_{R,\epsilon} - (0,\epsilon)$ and the Green's functions relate as follows:

Lemma 4.6 *The Green's function has the following scaling property:*

$$
R^{n-1}G^{R,\epsilon}[(x,t),(\xi,\tau)] = G^{1,0}[(R^{-1}(x,t-\epsilon), R^{-1}(\xi,\tau-\epsilon))]. \tag{6}
$$

Proof. By uniqueness it suffices to see that $R^{-(n-1)}G^{1,0}[R^{-1}(x,t-\epsilon), R^{-1}(\xi,\tau-\epsilon)]$ is indeed the Green function for the region $T_{R,\epsilon}$ at the point (ξ,τ). It is clear that it equals zero on $\partial T_{R,\epsilon}$. Furthermore for test function f we have

$$\int\int_{T_{R,\epsilon}} \Delta_{x,t} R^{-(n-1)} G^{1,0}[R^{-1}(x,t-\epsilon), R^{-1}(\xi,\tau-\epsilon)] f(x,t)dxdt$$
$$= \int\int_{T_{1,0}} \Delta_{y,s} G^{1,0}[(y,s), R^{-1}(\xi,\tau-\epsilon)] f(Ry, Rs+\varepsilon)dyds$$
$$= -f(\xi,\tau)$$

We did a substitution $(x,t) = (Ry, Rs+\epsilon)$. Note that there is R^{-2} term arising from the switch of $\Delta_{x,t}$ to $\Delta_{y,s}$ and R^{n+1} from the determinant. *QED*

Recall that the vector v maps each $T_{R,\epsilon}$ into a compact $K = K_{R,\epsilon} \subset \mathcal{D}_Q$. For technical reasons we have to exhaust the upper half space by compacts. We fix any compact set M in the open upper half space and consider R large enough and ϵ small enough so that $M \subset T_{R,\epsilon}$.

Let us start to use the size estimate of our Bellman function to obtain an estimate of the function value $b_{K,Q}(0, R+\epsilon)$ from above:

$$\begin{aligned}
& b_{K,Q}(0, R+\epsilon) \\
& \leq C\tilde{Q}_2(P_{R+\epsilon}|f|^2\omega^{-1}(0) + P_{R+\epsilon}\|\vec{g}\|^2\omega(0)) \\
& = c_n C\tilde{Q}_2 \int_{\mathbb{R}^n} |f|^2(y)\omega^{-1}(y)\frac{R+\epsilon}{((R+\epsilon)^2+|y|^2)^{\frac{n+1}{2}}}dy \\
& \quad + \int_{\mathbb{R}^n} \|\vec{g}\|^2(y)\omega(y)\frac{R+\epsilon}{((R+\epsilon)^2+|y|^2)^{\frac{n+1}{2}}}dy \\
& \leq c_n(R+\epsilon)^{-n}C\tilde{Q}_2(\|f\|^2_{\omega^{-1}} + \|\vec{g}\|^2_{\omega}).
\end{aligned}$$

For an estimate from below, Green's formula applied to our situation gives:

$$\begin{aligned}
& b_{K,Q}(0, R+\epsilon) \\
& = -\int\int_{T_{R,\epsilon}} G^{R,\epsilon}((x,t),(0,R+\epsilon))\Delta_{x,t} b_{K,Q}(x,t)dxdt \\
& \quad - \int_{\partial T_{R,\epsilon}} b_{K,Q}(x,t)\frac{\partial G^{R,\epsilon}((x,t),(0,R+\epsilon))}{\partial n}dxdt \\
& \quad + \int_{\partial T_{R,\epsilon}} G^{R,\epsilon}((x,t),(0,R+\epsilon))\frac{\partial b_{K,Q}((x,t))}{\partial n}dxdt
\end{aligned}$$

The first boundary term is negative because b is non-negative and the outward normal of the Green's function is negative on the boundary of $T_{R,\epsilon}$. The second boundary term vanishes because $G^{R,\epsilon} = 0$ on the boundary. So we have the following estimate:

$$
\begin{aligned}
&b_{K,Q}(0,R+\epsilon)\\
&\geq -\int\int_{T_{R,\epsilon}} G^{R,\epsilon}((x,t),(0,R+\epsilon))\Delta_{x,t}b_{K,Q}(x,t)dxdt.\\
&\geq -\int\int_{M} G^{R,\epsilon}((x,t),(0,R+\epsilon))\Delta_{x,t}b_{K,Q}(x,t)dxdt.
\end{aligned}
$$

since $-\Delta b \geq 0$ and where we recall that $M \subset T_{R,\epsilon}$. We continue the estimate using the scaling properties of the Green functions (6).

$$
b_{K,Q}(0,R+\epsilon) \geq -\int\int_M R^{-(n-1)}G^{1,0}(R^{-1}(x,t-\epsilon),(0,1))\Delta_{x,t}b(x,t)dxdt.
$$

Since $G^{1,0}((R^{-1}x,0),(0,1)) = 0$ we have

$$
\begin{aligned}
&b_{K,Q}(0,R+\epsilon)\\
&\geq -\int\int_M R^{-(n-1)}\Big\{G^{1,0}((R^{-1}x,R^{-1}(t-\epsilon),(0,1))-\\
&\qquad G^{1,0}((R^{-1}x,0),(0,1))\Big\}\Delta_{x,t}b(x,t)dxdt\\
&= -\int\int_M R^{-(n-1)}\frac{\partial G^{1,0}}{\partial t}(R^{-1}x,\tau)R^{-1}(t-\epsilon)\Delta_{x,t}b_{K,Q}(x,t)dxdt\\
&= -\int\int_M R^{-n}\frac{\partial G^{1,0}}{\partial t}(R^{-1}x,\tau)\Delta_{x,t}b_{K,Q}(x,t)dx(t-\epsilon)dt,
\end{aligned}
$$

where $0 \leq \tau \leq R^{-1}(t-\epsilon)$. Pulling this all together with the estimate from above,

$$
\begin{aligned}
&-\int\int_M R^{-n}\frac{\partial G^{1,0}}{\partial t}(R^{-1}x,\tau)\Delta_{x,t}b_{K,Q}(x,t)dx(t-\varepsilon)dt\\
&\leq b_{K,Q}(0,R+\epsilon) \leq c_n(R+\epsilon)^{-n}C\tilde{Q}_2(\|f\|^2_{\omega^{-1}}+\|\vec{g}\|^2_{\omega}),
\end{aligned}
$$

hence

$$
-\int\int_M \frac{\partial G^{1,0}}{\partial t}(R^{-1}x,\tau)\Delta_{x,t}b_{K,Q}(x,t)dx(t-\epsilon)dt \leq c_nC\tilde{Q}_2(\|f\|^2_{\omega^{-1}}+\|\vec{g}\|^2_{\omega})
$$

uniformly with respect to R and ϵ, for all given M. When R goes to infinity, the normal derivative $\frac{\partial G^{1,0}}{\partial t}(R^{-1}x,\tau)$ tends to $\frac{\partial G^{1,0}}{\partial t}(0,0)$ uniformly with respect to $(x,t) \in M$. But we know that the normal derivative $\frac{\partial G^{1,0}}{\partial t}(0,0)$ is exactly the normalizing factor c_n of the Poisson kernel (see [3] and the references therein Couhlon-Duong). Letting R go to infinity and ε go to zero yields for all compact M of the upper half space:

$$
-\int\int_M \Delta_{x,t}b_{K,Q}(x,t)dxtdt \leq C\tilde{Q}_2(\|f\|^2_{\omega^{-1}}+\|\vec{g}\|^2_{\omega}).
$$

Finally, letting M exhaust the upper half space establishes (5).

5 The comparison of classical and Poisson characteristic.

In this section we prove Theorem 3.2. We provide an example that demonstrates that $\tilde{A}_2 \neq A_2$ if $n > 1$. For the case $n = 1$, it is known that the two classes are the same. Infact, for $n = 1$ the estimates

$$\tilde{Q}_2(\omega) \lesssim Q_2(\omega) \lesssim \tilde{Q}_2(\omega)^2$$

are proven in [5]. If $n > 1$ however, an easy example shows that the Poisson integral of a simple power weight diverges, although the weight belongs to classical A_2. Consider $\omega_\alpha(x) = |x|^\alpha$. It is well known and straightforward to check that $\omega_\alpha \in A_2 \leftrightarrow |\alpha| < n$. Also $Q_2(\omega_\alpha) \sim \frac{1}{n^2-\alpha^2}$. We show that the Poisson integral $P_t w_\alpha(0)$ diverges for α close to n.

$$\begin{aligned}
&P_t(\omega_\alpha)(0) \\
&\sim \int_{\mathbb{R}^n} \frac{t}{(t^2+|x|^2)^{\frac{n+1}{2}}}|x|^\alpha dx \\
&= |S| \int_0^\infty \frac{t}{(t^2+r^2)^{\frac{n+1}{2}}} r^{\alpha+n-1} dr \\
&= |S| \sum_{k=1}^\infty \int_{2^{k-1}t}^{2^k t} \frac{t}{(t^2+r^2)^{\frac{n+1}{2}}} r^{\alpha+n-1} dr \\
&\gtrsim |S| \sum_{k=1}^\infty 2^{k-1} t \frac{t}{(t^2+2^{2k}t^2)^{\frac{n+1}{2}}} (2^{k-1}t)^{\alpha+n-1} \\
&\gtrsim t^\alpha |S| 2^{-\alpha-n-1} \sum_{k=1}^\infty 2^{(\alpha-1)k}
\end{aligned}$$

We see that this sum converges if and only if $\alpha - 1 < 0$. If $n \geq 2$ we have $w_\alpha \in A_2$ if and only if $|\alpha| < n$ so we can easily pick a valid α for which the above sum diverges.

Thus not every weight in A_2 is in $\tilde{A}_2$. The converse is still true, though. Let $w \in \tilde{A}_2$, and let B be a ball with center a and radius r. Then for $y \in B$, $|a-y| < r$, and so

$$\frac{1}{r^n} \leq 2^{\frac{(n+1)}{2}} \frac{r}{(r^2+|a-y|^2)^{\frac{n+1}{2}}}$$

and so

$$\langle \omega \rangle_B \leq C \int_B \frac{r \;\, w(y)}{(r^2+|a-y|^2)^{\frac{n+1}{2}}} dy \leq C' P_r(\omega)(a),$$

and similarly for $\langle \omega^{-1} \rangle_B$. Thus $Q_2(\omega) \leq \tilde{C} \tilde{Q}_2(\omega)$.

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